irreguläre fraktale

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In nature there are shapes that are very similar to the fractals defined in mathematics. For example, in **Romanesco Broccoli**, structures of **self-similarity** can be found in multiple repetitions.

Romanesco Brokkoli

https://en.wikipedia.org/wiki/Romanesco_broccoli#/media/File:Fractal_Broccoli.jpg

This property is also found in ferns and many other objects.

Fractals in mathematics have the following properties self-similarity, often scale invariance and a broken Hausdorff dimension. In Romanesco Broccoli, self-similarity and scale invariance are evident in the fact that the conical bulges appear again and again in smaller forms. With mathematical fractals these properties can be exactly fulfilled.

A quote: ". . . fractal also means irregular and this describes the difference of the Euclidean regular geometric structures to self-similar structures. . . "

From: COMPUTER + UNTERRICHT, issue 14/1994, CHAOS / FRAKTALE Erhard Friedrich Verlag, http://www. fraktalwelt. de/extra/dfw.pdf

But this kind of irregularity is not the one discussed in the present paper.. Rather, the term is derived from the irregular functions described below.

A mathematical fractal of the plane is the Sierpinski triangle. It is formed by cutting a similar triangle with half the side length from an initial triangle using the method shown below. With smaller and smaller triangles this process is repeated infinitely.
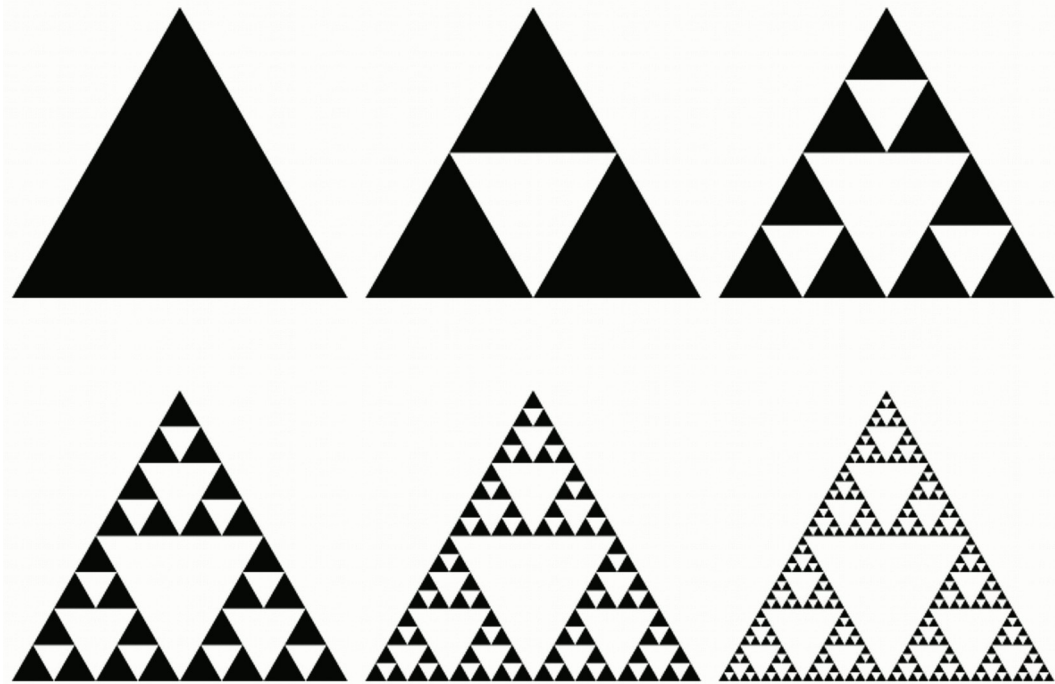
Sierpinski-Dreieck

The area of the fractional remainder tends towards zero. The Latin word "frangere" - "to break" - thus leads to the term fractal.

After an infinite number of extractions of triangles, only a linear structure would remain. In geometry, lines are one dimensional. The increasingly complex branching suggests another concept of dimension, the Hausdorff dimension, which will be presented here in simplified form. The structure is covered with circles of equal size, and the minimum number of circles with radius R, needed for covering the structure is chosen. This number is called N(R). The smaller the radius R becomes, the larger N becomes.

$$N(R) \sim \frac{1}{R^D}$$

If the radii get smaller and smaller so that they tend towards zero, then D in this formula gives the Hausdorff dimension.

The Sierpinski Triangle has the Hausdorff dimension: D ~= 1. 5849. . . . For objects of conventional geometry, the Hausdorff dimension is equal to the geometric dimension. Hausdorff dimensions can have values that are not natural numbers.
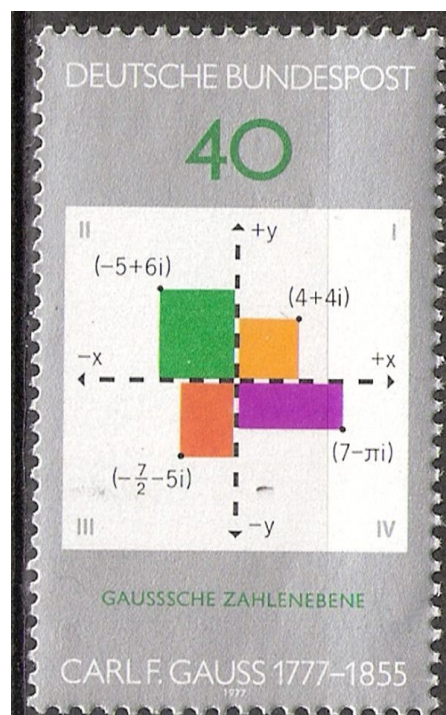
The complex numbers

In order to create images, an area is needed. For this purpose, one uses the body of complex numbers - the Gaussian number plane. Benoît Mandelbrot generated fractals by iterative calculations with numbers of the complex number plane. The Mandelbrot set is a fractal that is often called "Apfelmännchen" in German.
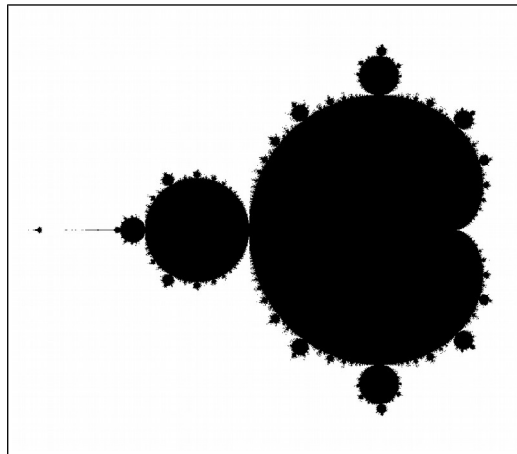
Apfelmännchen

How do you get complex numbers? You need another dimension in addition to the straight line of the real numbers. Since the equation $x^2 = -1$ has no solution in the real numbers domain, because the square of real numbers is never negative, imaginary numbers are introduced with **i** being the number with the property: $i^2 = -1$. The real multiples of the number **i** form the imaginary numbers.

The imaginary numbers can be represented on an axis that passes perpendicularly through the origin of the real axis. The complex numbers are located on the resulting plane and are defined by two coordinates, the real part **x** and the imaginary part **y**.

Iteration with complex functions:

In the following, the numbers z and c will be defined as complex numbers. For example, a nonlinear iteration process is defined by $z_{n+1} = z_n^2 + c$. $z_0=0$ is set as initial value. A point c is selected from the complex number plane and for this c, the sequence z_n is calculated gradually. For $c=0$, the sequence always remains at $z_n=0$, for $c=1$, however, the sequence remains on the real number line, but with **1, 2, 5, 26, 677…** it quickly grows beyond all limits. For some starting points c the elements of the sequence remain within a fixed circle around the origin, for others they do not. Depending on this behavior of the sequence, point c on the plane can be colored black or white. Thus the picture of the Mandelbrot set shown above is created. The iteration function $f(z) = z_n^2 + c$ used here is quadratic and not linear. The iteration of linear functions does not yield fractals, only boring smooth structures.
The function $f(z)=z^2$ is complex differentiable, the derivative is:
$f'(z) = 2*z$. The complex differentiable functions are also called analytical, holomorphic or regular (old designation).

Other regular functions are for example:

$f(z) = e^z$, $f(z) = sin(z)$, $f(z) = cos(z)$, $f(z) = tan(z)$, $f(z) = cot(z)$,   and many others.

The regular function $f(z) = z_n^2$ can be split into its real and imaginary parts:
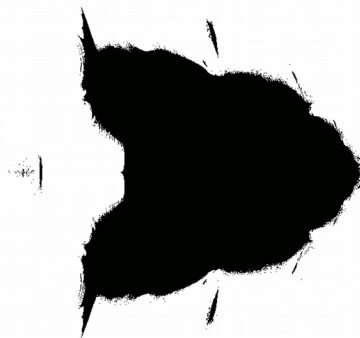
$re(f(z)) = x^2 - y^2$
$im(f(z)) = 2*x*y$

If we now form a similar function g(z) in the following way:

$re(g(z)) = x^2 - y^2$
$im(g(z)) = 2*x*y + y$

then a closed representation as g(z) without explicit use of the real and imaginary part is not possible for this function. The addition of **y** in the imaginary part makes this function an irregular function. It is nonholomorphic, non-analytic.


attraction area of g(z)

Fractals that are created by iteration of irregular functions are called irregular fractals here.

Method of coloring

According to the iteration calculation, there is an iteration number for each point of the rectangle that belongs to the image. This number indicates how many steps were necessary to force the numerical value out of a circle with a previously defined radius around the origin. If after 100 steps the circle has not yet been left, then it is assumed that the iteration keeps the point within an attraction area. These points on the rectangle are assigned the numerical value 0 and are colored black. For the irregular function g(z) defined above, the following representation is obtained by plotting the calculated iteration number for each point of the rectangle as a rod perpendicular to the plane of the rectangle.
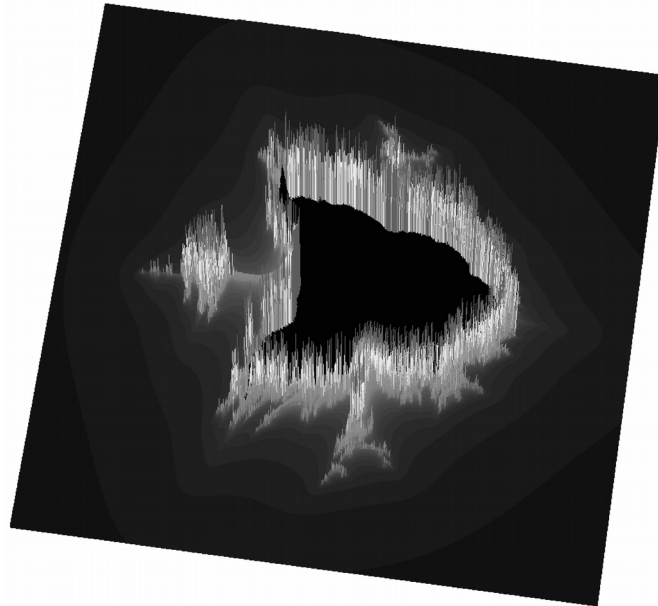

Area of g(z) in the 3-d representation

As a "normal observer" for color vision, humans have three types of receptors on the retina in their eyes, one for red (R), one for green (G) and one for blue (B). Therefore, the physiology of colour vision can be described as three-dimensional. For the eye, colour vision is additive, i. e. Mixed colours such as yellow are obtained by additive superimposition of the primary colours red and green. Therefore, the RGB system is used for the representation of colours in self-luminous systems, such as a TV screen, a computer monitor or a projector:
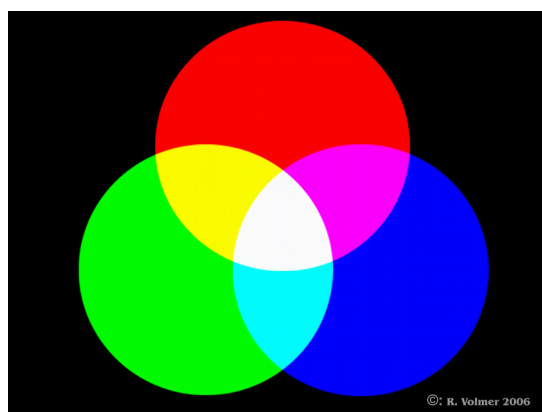
Additive colour mixing in the RGB system

The corresponding RGB colour space is represented as a cube in the three-dimensional coordinate system. The axes are divided into 255 intervals, the end points are 0 and 255.
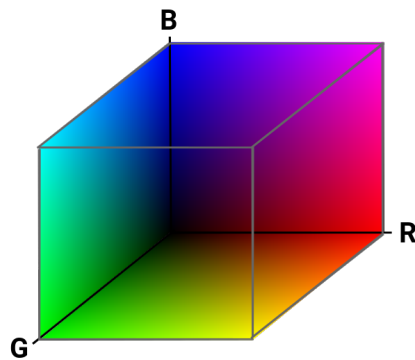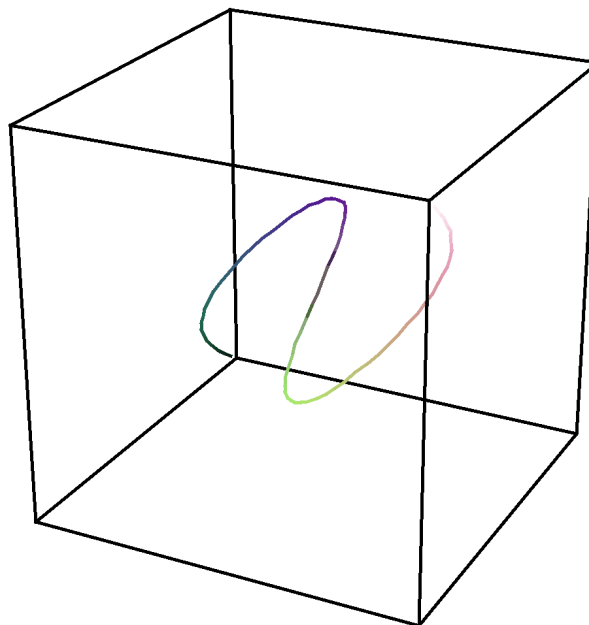

RGB color space

In this color space, a path is defined as a mathematical curve that traverses the colors to be used for the fractal. In the standard case, iteration is performed for pixels up to a number of 100. The iteration number 0 means that the point is to be colored black. On the way, depending on the number of iterations, the colors are gradually read off from 1 to 100 and assigned to the point in the fractal rectangle.

The following figure shows the color path in the RGB color space that is traversed by the functions

R = x
G = sin(x*3.1415926*4)*70
B = sin(x*3.1415926*2)*70

Color path in the RGB color space

The iteration number of the point on the rectangle to be colorized is entered as x in these formulas. The three values R, G and B can then be calculated, which define the color to be used for this point.

This method is called the parameterization of a path. Alternatively, the colors that have been passed through on the way can be displayed linearly:
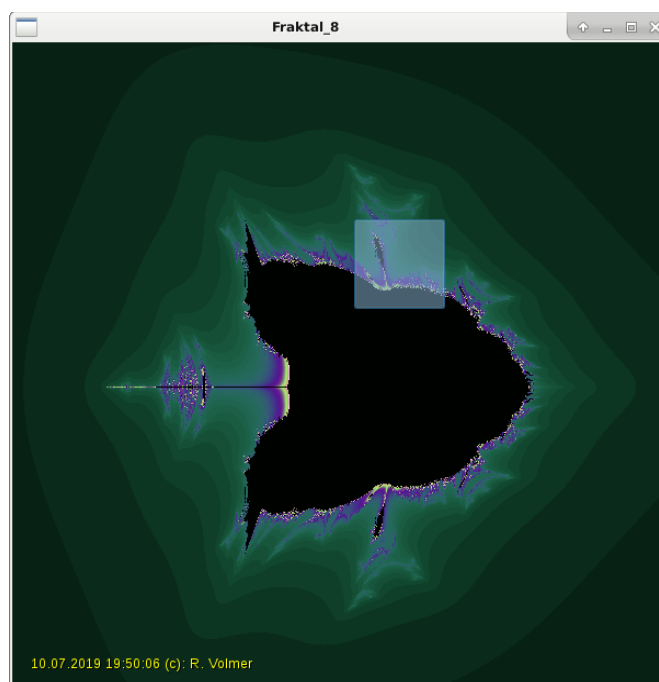

Colors passed along the color path

The fractal colored with this parameterization then has the following appearance:
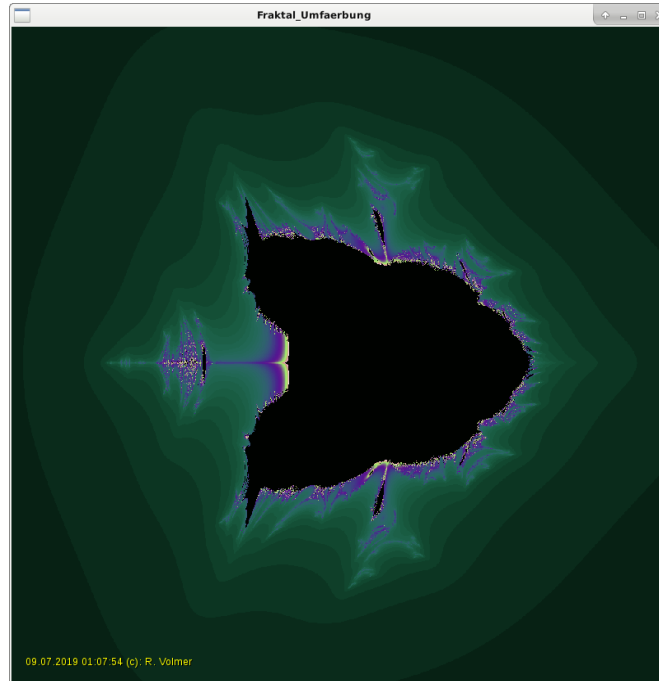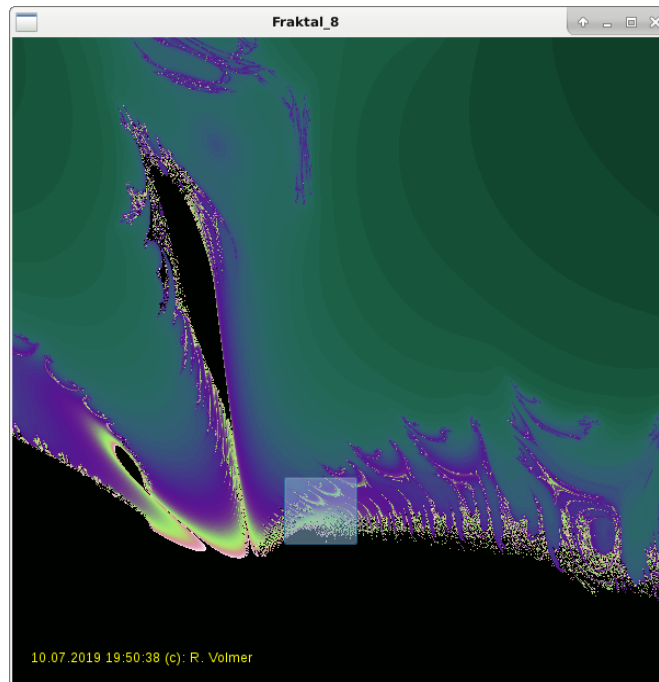

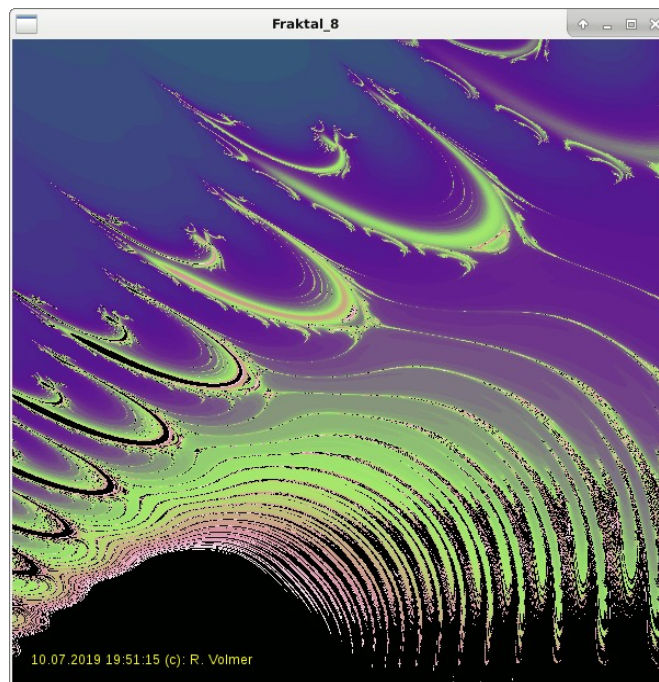**Selection of an interesting area in a fractal**

You can see that the interesting areas are usually located between the attraction area (black) and the divergence area (colored). The mouse pointer is used to select a rectangle to be enlarged.

A recalculation for the enlarged area leads to the image below:

And another enlargement yields the following:

This image can then be saved as a final version in a higher resolution.

For illustrations that do not originate from the author, the origin is indicated.

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